Chaotic phase similarities and recurrences in a damped-driven Duffing oscillator

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We report strong evidence of remarkably close periodic repetitions of the structuring of the parameter space of a damped-driven Duffing oscillator as the amplitude of the drive increases. Families of period-adding cascades and some intricate networks of periodic oscillations embedded in chaotic phases are also found to recur closely as the driving force grows. Such surprising regularities suggest that some hitherto unknown renormalization mechanism may be operating in higher codimension, controlling the alternation of chaos and order in parameter space of certain flows.

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The ubiquitous presence of period-doubling cascades and alternations of periodic-chaotic windows are nowadays familiar features that have been observed experimentally in virtually all sorts of natural phenomena, particularly in physics and engineering [1–7]. A fair understanding of these phenomena was obtained in the past two decades by studying certain paradigmatic discrete-time dynamical systems like, e.g., the circle and Hénon maps. The quantification of the bifurcation structuring in discrete-time dynamical systems is of practical utility for applications ranging from all sorts of mechanical devices and models to secure communications with lasers [1–7].

Discrete-time maps are particularly valuable models because difference equations directly express future states in terms of present ones, and obtaining chronological sequences of points poses no computational difficulties. Parameter space diagrams illustrating the intricate structure of *chaotic phases* for maps were obtained quite early [8,9]. In contrast, for continuous-time flows the differential equations must first be solved, usually by approximating them by numerical methods. The simplicity of iterating maps together with a basic knowledge attesting that similar results hold both for maps and for flows led to an exhaustive exploration of discrete mappings, while the equivalent problem for continuous flows still awaits investigation. There is much to explore in parameter space of the computationally much more demanding problem of flows, even for elementary situations involving the simultaneous variation of just two parameters.

A pressing problem that remained virtually unexplored is the one concerning the *inner structuring* of the *chaotic phases* [10] so abundant in flows. Of course one knows about alternations of periodic and chaotic windows in systems ruled by differential equations. But a systematic multiparameter investigation of the structuring of the chaotic phases of flows is just starting. Some aspects of the intricate organization existing inside individual chaotic phases were described by us recently for lasers and other experimentally accessible flows [11–13]. The aim of this paper is to provide strong numerical evidence that chaotic phases together with their inner structural organization and intricate bifurcation sets *repeat periodically almost isomorphically over extended portions of parameter space* for the Duffing oscillator, as may be recognized from Figs. 1–3.

The Duffing oscillator is a prototypical example of an externally driven nonlinear oscillator where deterministic chaos was detected no later than 1961 [14–16]. While repetitions of bifurcation curves for some low-periodic orbits were observed earlier in the Duffing oscillator [17] and in some other periodically driven systems [18], so far such repetitions were observed only for a few of the lowest periods, not for chaotic phases. In fact, most of the earlier phase diagrams do not even mention the occurrence of chaotic phases and only relatively seldom are the external boundaries of chaotic phases delimited in parameter space. In contrast, the central aim of this work is to determine where chaotic phases occur and to present a detailed description of

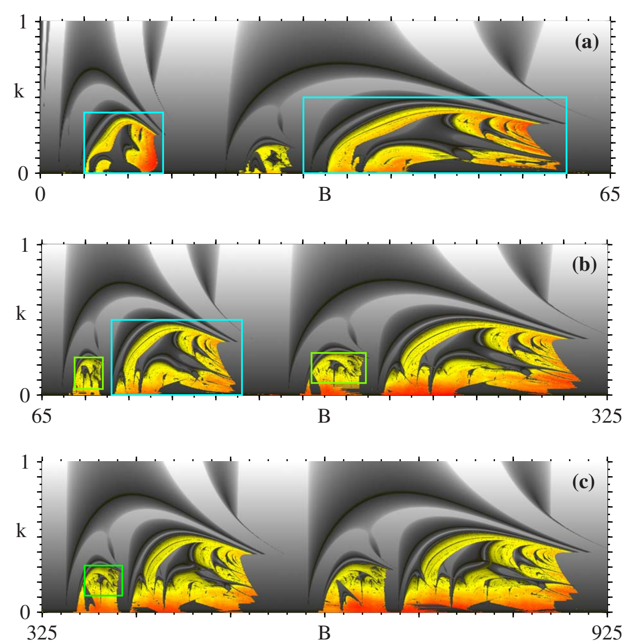

FIG. 1. (Color online) Phase diagrams displaying sequences of chaotic regions (colors; lighter shadings) which recur as the driving amplitude B of the Duffing oscillator increases. These phase diagrams discriminate between *regularity*, indicated by the darker shadings, corresponding to negative Lyapunov exponents, and *chaos*, indicated by the colors (lighter shadings), corresponding to positive exponents. The sequences of chaotic regions contained inside the two sets of boxes are magnified in Figs. 2 and 3.

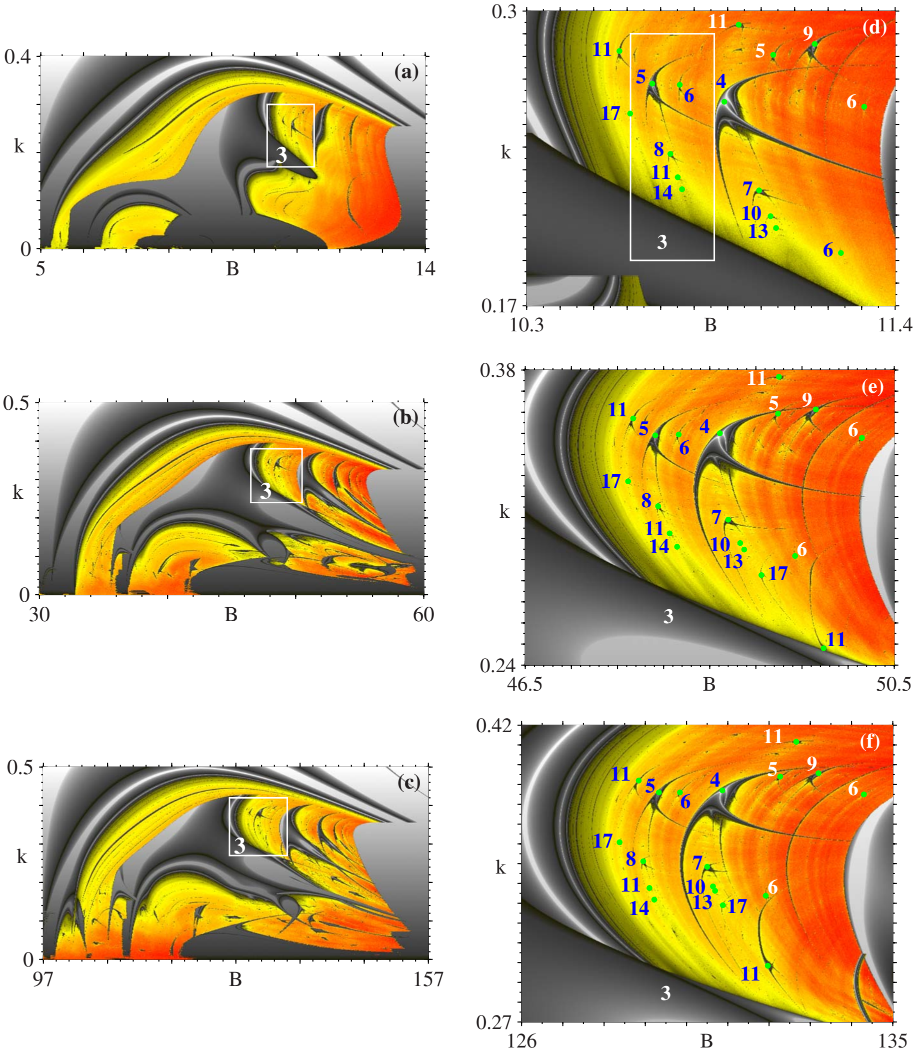

FIG. 2. (Color online) Details of the chaotic phases contained inside the blue boxes in Fig. 1 displaying structural similarities as the forcing amplitude B increases. The right side shows magnified views of the white boxes on the left. Darker islands on the right are *shrimps* [20]. Numbers refer to the period of individual shrimps in multiples of 2π, the period of the external drive. Distinct period-adding cascades converge toward the accumulation horizons at the boundary of the dark zone of period 3 in the lower left corner. The right side illustrates the great similarity of structuring as B grows. Details inside the box in (d) are shown magnified in Fig. 4. The coloring scheme is the same as described in Fig. 1. Note differences in scales.

the inner structuring of the regularities embedded in them. The chaotic phases and the regular structuring embedded in them are the key objects studied in this work [10].

For increasing parameter resolution, Figs. 1–3 provide compelling evidence of sequences of remarkably close repetitions of the structuring of wide portions of the chaotic phases of the damped-driven Duffing oscillator [14–16,19], a prototypical illustrative flow, as the amplitude of the driving force grows. By "close repetitions" we mean that, modulo a few specific details, it is possible to recognize wide parameter regions which are essentially isomorphic copies of each other. This type of recurrent behavior is illustrated in Fig. 1. In this and all subsequent figures, colors are used to indicate chaotic phases (positive Lyapunov exponents) while gray shadings mark periodic oscillations (negative exponents). The red tonalities denote positive exponents with larger magnitudes.

Recurrences analogous to those seen in Fig. 1 exist also at finer scales of resolution. Additionally, there is also a repetition of a number of finer details and structurings inside sequences of chaotic phases like, e.g., the organization of intricate networks of regular phases corresponding to periodic oscillations. Note that while recent works described regularities inside *single* chaotic regions [11–13], here we report close structural repetitions among long *series* of distinct chaotic regions. This fact strongly suggests that a generalized type of renormalization group might be controlling the chaos-order alternation over wide parameter surfaces in flows and, in particular, might be aligning periodicity islands, like the *shrimp* [20] along very specific *directions* [8,9] in control space. We mention that promising two-parameter renormalization group analysis in two-dimensional space have been obtained for Mandelbrot sets [21]. Similar results should hold in more general settings [22]. But these scenarios seem to be considerably simpler than those reported here.

Among the several models investigated originally by Duffing [19] we select one that is arguably the simplest, viz., the single-well oscillator defined by the expression

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + x^3 = B\cos t. \quad (1)$$

Here, k controls the damping while B controls the amplitude of an external periodically varying driving force. Chaotic solutions of Eq. (1) were discovered as early as 1962–63 [14–16]. Duffing's equation has since become the subject of extensive literature and a few surveys of its most fundamental properties exist [1,15,23–25]. All oscillators considered

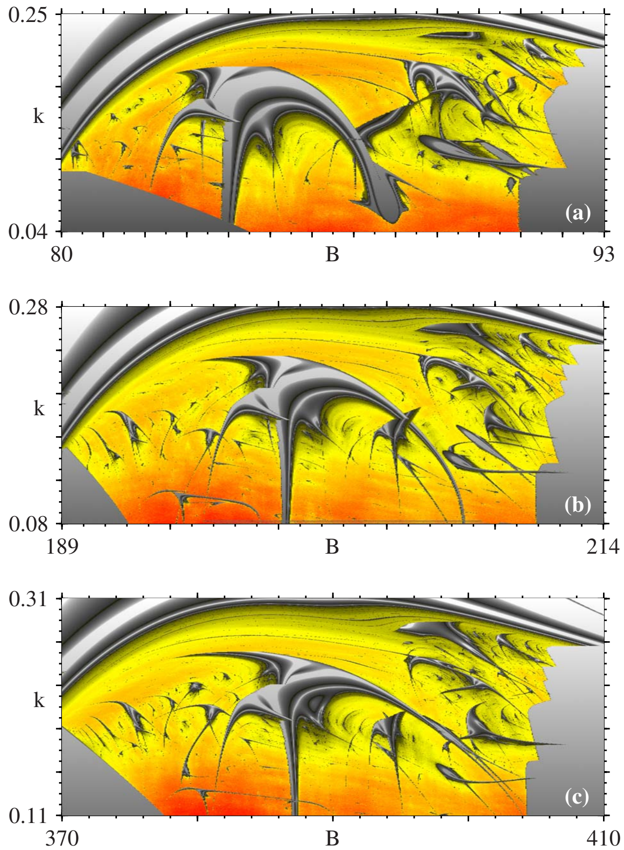

FIG. 3. (Color online) Zoom of the green boxes of Fig. 1 evidencing complex bifurcation sets embedded in chaotic phases with close structural similarity as B grows. The inner structuring of this *sequence of chaotic regions* is very distinct from the sequence in Fig. 2. Both sequences are typical of flows. The coloring is the same as described in Fig. 1.

by Duffing [19] involve dissipation and cubic stiffness and are nowadays popular test grounds where to investigate intricate dynamical changes induced by periodically varying forces [14–16,23–30]. A particularly interesting application concerns the mechanical bistability of atoms cooled in an optical lattice [31].

The nature of the solutions of Eq. (1) up to $B=23.5$ has been described in a comprehensive atlas by Ueda [28]. Subsequently, Byatt-Smith [32] described the solutions up to $B=1000$ while Robinson [33] extended further the phase-space analysis up to $B=5000$. Results for different normal forms of Duffing's single-well oscillator exist [14–17]. We reiterate: Regularities among curves delimiting domains of some low-periodic and chaotic solutions may be found in the literature [14–18] along with some interesting theoretical approximations for them [34]. But a multiparametric study of the inner structuring of chaotic phases is still an open problem, a problem that we wish to focus here.

Similarly as done in Ref. [28], Robinson corroborated his numerical integrations by comparing them with experimental results. From his analysis of the time evolution of the experimental patterns Robinson concluded many things. As of particular interest in the present context (and obviously in an *ad hoc* manner and with the benefit of hindsight), we quote the following observation (using our notation for parameters): "Although the details of [the] pattern are different for other ranges of B, markedly different if k is decreased, certain features of the pattern always repeat as B is increased from 0 to 5000."

Extended repetitions of structurings, either periodic or chaotic, turn out to be dominating characteristics of the oscillator when contemplated in parameter space. We discovered this while computing systematically a series of high-resolution parameter space diagrams or, equivalently, $k-B$ charts [15], or phase diagrams, like those in Figs. 1–3. Such diagrams were obtained by computing all three Lyapunov exponents for Eq. (1) with a fixed-step fourth-order Runge-Kutta integrator.

In the phase diagrams depicted in Figs. 1–3, each individual panel displays in colors the numerical value of highest nonzero Lyapunov exponent computed over equally spaced parameter meshes with 600×600 exponents. Individual figures display exponents using two independent color scales. A red-yellow scale was used to represent phases containing *chaotic oscillations*, i.e., positive exponents (red indicating exponents of larger magnitude), and a black-white gray scale for *periodic oscillations*, i.e., negative exponents, with white indicating the more negative exponents. Black was used as a separator of both coloring schemes in order to mark the locus of zero exponents where bifurcations occur. Figures were generated sweeping parameters horizontally from left to right. Each new horizontal line was started always from a fixed initial condition, $(x,\dot{x})\equiv(0.5,0.1)$, chosen arbitrarily. Then, we increased B from this initial value by *following the attractor*, i.e., by starting each additional computation of Lyapunov exponent from the value $(x,\dot{x})$ that was in the memory of the computer when the computation of the previous values of B was ended. Of course, in this way we loose information about the coexistence of multiple attractors. Comparing figures generated scanning parameters along distinct directions one learns to recognize the regions more likely to contain multistability, although one can only be sure of it after computing explicitly basins of attraction for parameters in such regions. Since to map attractor coexistence is an even more time-consuming task than computing phase diagrams, we postpone it for a future work.

The three panels in Fig. 1 compare the parameter structuring when the amplitude B of the forcing increases. In these panels one clearly sees a systematic repetition of structuring, particularly of the colored shadings used to represent chaotic phases. As B grows there is a regular repetition of two sequences of chaotic phases which are contained in the larger (blue) and in the smaller (green) sequence of boxes. The larger blue boxes are magnified in Fig. 2 and the green ones in Fig. 3. The color coding defined for Fig. 1 is the same used for all similar looking figures.

Given the remarkable agreement of the structuring in Fig. 1, it is natural to inquire about the repetition of details at finer scales. This situation is illustrated in Figs. 2 and 3. The left side in Fig. 2 shows magnified views of the larger boxes in Fig. 1, while the right side in Fig. 2 shows additional magnifications of the boxes on its left. The specific parameter windows on the right side were selected just because they also contain accumulations analogous to the ones found recently in semiconductor lasers and other flows [12,13]. But there is a plethora of additional windows displaying analogous structural similarities.

Numbers in Fig. 2 indicate the periodicity of the solutions in multiples of 2π, the driving period, for parameters located in the dark islands embedded in chaotic phases. Whenever islands are too small to be easily seen in the scale of the figure, a green dot was used to mark their position. Looking at the right side we recognize that the structural organization around the two largest islands of periods 4 and 5 is quite similar in all three panels. It is also apparent that the agreement is larger when B increases, a tendency that remains true at larger values of B, as one sees from additional figures that could not be included here. This is corroborated by Fig. 3, showing magnifications of the smaller boxes in Fig. 1.

One characteristic feature that recurs in the right panels in Fig. 2 is the presence of series of period-adding cascades. For instance, moving downward from the larger period-4 islands located near the center of the three panels it is possible to recognize two adding cascades when decreasing k and increasing B:

$$4 \to 7 \to 10 \to 13 \to 16 \to \cdots \to 3,$$

$$5 \to 8 \to 11 \to 14 \to 17 \to \cdots \to 3.$$

As it is clear, in these cascades the period increases steadily by 3, which is the period of the large domain toward which both adding cascades accumulate [12].

Figure 4 shows the period-5 cascade located in the box of Fig. 2(d). With the exception of the first few periods of the cascade, all remaining periods emerge perfectly aligned along a specific direction in parameter space [8,9]. For the cascade in Fig. 4 this direction is well approximated by the straight line

$$k = -0.3972B + 4.4966. \tag{2}$$

In Fig. 4(b), the boundary between the chaotic phase and the period-3 domain may be roughly approximated by

$$k = -0.0700B + 0.9618. \tag{3}$$

These lines intersect at $(B,k) \simeq (10.803, 0.2056)$, the approximate location of the accumulation point of the adding cascade. This point is indicated by the larger red dot at the bottom of Fig. 4(b).

In summary, we reported a description of the isomorphic arrangement—with approximate *translation* and scale recurrences—of periodic regions embedded into chaotic ones in the two-dimensional control parameter space of a simple damped-driven Duffing oscillator. This close isomorphic repetition seems to indicate the possibility of relating recurring structures by affine or nearly affine [9,35] changes of parameter that would be interesting to investigate. It would be equally interesting to characterize the stretching factors ruling repetitions in Figs. 2 and 3 and to find a way of quantifying generically similarities involving the simultaneous variation of more than one parameter. Do *compact* windows of chaos (called "robust chaos" by Banerjee *et al.* [36]), which are so critical for secure communications with chaotic lasers, occur in parameter-space of models ruled by differential equations or are the chaotic phases of flows invariably riddled by windows of periodicity in all scales?

These tasks are rather computer-intensive but very likely to be rewarding for the understanding of the detailed structuring of chaotic phases and of sequences of chaotic phases in parameter space of flows. To conclude, we remark that the global recurrent arrangement of periodic and chaotic behaviors described here are not restricted to the periodically driven oscillator used but also show up in the parameter space of other flows, in particular in autonomous flows. This will be reported elsewhere.

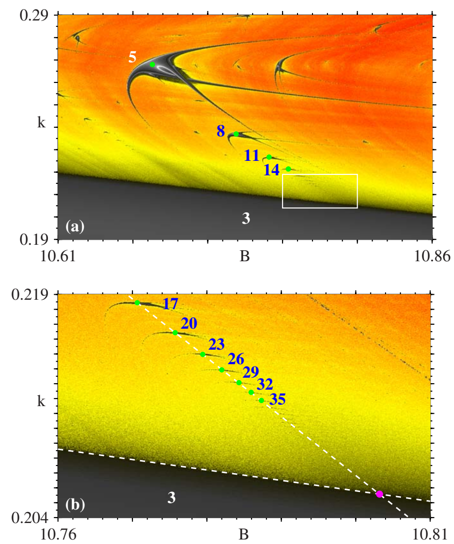

FIG. 4. (Color online) (a) Zoom of the accumulation horizon seen in Figs. 2(a) and 2(d). Numbers denote periods in multiples of 2π, the period of the external drive. (b) Magnification of the box in (a). The period-adding cascade indicated by the green dots accumulates along the upper dashed line, defined by Eq. (2), toward the red dot on the accumulation horizon, the lowest dashed line, defined by Eq. (3), boundary between positive and negative Lyapunov exponents. The coloring is the same as described in Fig. 1.

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